# Trigonometric Integrals

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When dealing with integrals involving trigonometric functions, the following formulas from precalculus are often helpful:

1. $\sin^2 x + \cos^2 x = 1$, $1 + \tan^2 x = \sec^2 x$, $1 + \cot^2 x = \csc^2 x$
2. $\sin x = \frac{1}{\csc x}$, $\cos x = \frac{1}{\sec x}$, $\tan x = \frac{1}{\cot x}$
3. $\tan x = \frac{\sin x}{\cos x}$, $\cot x = \frac{\cos x}{\sin x}$
4. $\sin 2x = 2 \sin x \cos x$, $\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x = 2\cos^2 x - 1$

**Example 1.** *Evaluate* $\int \cos^3 x\, dx$

$$\begin{aligned}\int \cos^3 x\, dx &= \int \cos^2 x\, d\sin x \\ &= \int 1 - \sin^2 x\, d\sin x \\ &= \sin x - \frac{1}{3}\sin^3 x + C\end{aligned}$$

**Example 2.** *Evaluate* $\int_0^\pi \sin^2 x\, dx$

$$\begin{aligned}\int_0^\pi \sin^2 x\, dx &= \int_0^\pi \frac{1 - \cos 2x}{2}\, dx \\ &= \frac{1}{2}\int_0^{2\pi} \frac{1 - \cos u}{2}\, du \\ &= \frac{1}{4}(\int_0^{2\pi} 1\, du - \int_0^{2\pi} \cos u\, du) \\ &= \frac{\pi}{2}\end{aligned}$$

**Example 3.** *Evaluate* $\int \sin^3 x \cos^2 x\, dx$

$$\begin{aligned}
\int \sin^3 x \cos^2 x\, dx &= -\int \sin^2 x \cos^2 x\, d\cos x \\
&= -\int (1-\cos^2)\cos^2 x\, d\cos x \\
&= \int \cos^4 x - \cos^2 x\, d\cos x \\
&= \frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x + C
\end{aligned}$$

**Example 4.** *Evaluate* $\int \cos^4 x\, dx$

$$\begin{aligned}
\int \cos^4 x\, dx &= \int (\frac{1+\cos 2x}{2})^2\, dx \\
&= \frac{1}{4}\int \cos^2 2x + 2\cos 2x + 1\, dx \\
&= \frac{1}{4}\int \frac{1+\cos 4x}{2} + 2\cos 2x + 1\, dx \\
&= \frac{1}{4}\int \frac{1}{2}\cos 4x + 2\cos 2x + \frac{3}{2}\, dx \\
&= \frac{1}{8}\int \cos 4x\, dx + \frac{1}{2}\int \cos 2x\, dx + \frac{3}{8}\int 1\, dx \\
&= \frac{1}{32}\int \cos 4x\, d4x + \frac{1}{4}\int \cos 2x\, d2x + \frac{3}{8}x \\
&= \frac{1}{32}\sin 4x + \frac{1}{4}\sin 2x + \frac{3}{8}x + C
\end{aligned}$$

Sometimes we will make use of trigonometric functions to do some integration by substitution. Some general idea is the following:

$$\begin{aligned}
&\sqrt{a^2-x^2} \text{ let } x = a\sin t, -\frac{\pi}{2} \le t \le \frac{\pi}{2} \\
&\sqrt{a^2+x^2} \text{ let } x = a\tan t, -\frac{\pi}{2} < t < \frac{\pi}{2} \\
&\sqrt{x^2-a^2} \text{ let } x = a\sec t, 0 \le t < \frac{\pi}{2} \text{ or } \pi \le t < \frac{3\pi}{2}
\end{aligned}$$

**Example 5.** *Evaluate* $\int \sqrt{4-x^2}\,dx$ *Let* $x = 2\sin t$*, where* $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$.

$$\begin{aligned}
\int \sqrt{4-x^2}\,dx &= \int \sqrt{4-4\sin^2 t}\,d2\sin t \\
&= \int 2\cos t\,d2\sin t \\
&= \int 4\cos^2\ dt \\
&= \int (1+\cos 2t)\,d2t \\
&= 2t + \sin 2t + C \\
&= 2\sin^{-1}\frac{x}{2} + \frac{x}{2}\sqrt{4-x^2} + C
\end{aligned}$$

**Example 6.** *Find the area enclosed by the ellipse*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

*When* $y \ge 0$, $y = \frac{b}{a}\sqrt{a^2-x^2}$*, so the area is*

$$2\int_{-a}^{a} \frac{b}{a}\sqrt{a^2-x^2}\,dx = \frac{2b}{a}\int_{-a}^{a} \sqrt{a^2-x^2}\,dx$$

*Let* $x = a\sin t$, $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$*, then*

$$\begin{aligned}
\frac{2b}{a}\int_{-a}^{a} \sqrt{a^2-x^2}\,dx &= \frac{2b}{a}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{a^2-a^2\sin^2 t}\,da\sin t \\
&= 2ab\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 t\,dt \\
&= 2ab\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1+\cos 2t}{2}\,dt \\
&= ab(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1\,dt + \frac{1}{2}\int_{-\pi}^{\pi} \cos u\,du) \\
&= ab\pi
\end{aligned}$$

**Example 7.** *Find* $\int \frac{1}{x^2\sqrt{x^2+4}}\,dx$

*Let* $x = 2\tan t$, $-\frac{\pi}{2} < t < \frac{\pi}{2}$, *then*

$$\begin{aligned}
\int \frac{1}{x^2\sqrt{x^2+4}}\,dx &= \int \frac{1}{(4\tan^2 t)2\sec t}\,d2\tan t \\
&= \frac{1}{4}\int \frac{\cos^2 t}{\sin^2 t} \times \cos t \times \frac{1}{\cos^2 t}\,dt \\
&= \frac{1}{4}\int \frac{\cos t}{\sin^2 t}\,dt \\
&= \frac{1}{4}\int \frac{1}{\sin^2 t}\,d\sin t \\
&= -\frac{1}{4\sin t} + C \\
&= -\frac{\sqrt{4+x^2}}{4x} + C
\end{aligned}$$

*The last equality follows from* $\frac{1}{\sin t} = \sec t = \sqrt{1+\cot^2 t} = \sqrt{1+\frac{4}{x^2}}$

**Example 8.** *Evaluate* $\int_0^{\frac{3}{10}} \frac{x^2}{\sqrt{9-25x^2}}\,dx$

*Let* $x = \frac{3}{5}\sin t$, *when* $x$ *is from* $0$ *to* $\frac{3}{10}$, $t$ *is from* $0$ *to* $\frac{\pi}{6}$

$$\begin{aligned}
\int_0^{\frac{3}{10}} \frac{x^2}{\sqrt{9-25x^2}}\,dx &= \int_0^{\frac{\pi}{6}} \frac{(\frac{3}{5}\sin t)^2}{\sqrt{9-9\sin^2 t}}\,d\frac{3}{5}\sin t \\
&= \int_0^{\frac{\pi}{6}} \frac{9}{125}\sin^2 t\,dt \\
&= \frac{9}{125}\int_0^{\frac{\pi}{6}} \frac{1-\cos 2t}{4}\,d2t \\
&= \frac{9}{125}\int_0^{\frac{\pi}{3}} \frac{1-\cos u}{4}\,du \\
&= \frac{9}{500}(\frac{\pi}{3} - \frac{\sqrt{3}}{2}) \\
&= \frac{3\pi}{500} - \frac{9\sqrt{3}}{1000}
\end{aligned}$$

**Example 9.** *Compute* $\int \frac{1}{\sqrt{x^2-a^2}}$

*Let* $x = a\sec t$, $0 < t < \frac{\pi}{2}$ *or* $\pi < t < \frac{3\pi}{2}$,

$$\begin{aligned}
\int \frac{1}{\sqrt{x^2-1}} = \int \frac{1}{\tan t}\, d\sec t \\
&= \int \frac{\cos t}{\sin t}\frac{\sin t}{\cos^2 t}\, dt \\
&= \int \frac{1}{\cos t}\, dt \\
&= \int \frac{1}{\cos^2 t}\, d\sin t \\
&= \int \frac{1}{1-\sin^2 t}\, d\sin t \\
&= \frac{1}{2}(\int \frac{1}{1-\sin t}\, d\sin t + \int \frac{1}{1+\sin t}\, d\sin t) \\
&= \frac{1}{2}(\ln(1+\sin t) - \ln(1-\sin t)) + C \\
&= \frac{1}{2}\ln\frac{1+\sin t}{1-\sin t} + C \\
&= \frac{1}{2}\ln\frac{(1+\sin t)^2}{(1-\sin t)(1+\sin t)} + C \\
&= \frac{1}{2}\ln\frac{(1+\sin t)^2}{\cos^2 t} + C \\
&= \ln|\frac{1+\sin t}{\cos t}| + C \\
&= \ln|\sec t + \tan t| + C \\
&= \ln|\frac{x}{a} + \sqrt{\frac{x^2}{a^2} - 1}| + C \\
&= \ln|\frac{x}{a} + \frac{\sqrt{x^2-a^2}}{a}| + C \\
&= \ln|x + \sqrt{x^2-a^2}| - \ln a + C \\
&= \ln|x + \sqrt{x^2-a^2}| + C
\end{aligned}$$